MBHS Economics Club

Modern Economic Thought Notes

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The Definition of a Cake

- A cake is defined to be a 1-D segment, a finite 2-D region, or any subset of R^n
- We require a few properties of our cake:
    1. A cake doesn't not have to be *homogenous*, and rather can be *heterogenous*, meaning different parts can have different values
    2. A cake is *divisible*, meaning you can cut it up into arbitrarily small pieces
- We will divide up our cake between **N** people, with person **i** receiving $\mathbf{P_i}$
- Our cake is $\mathbf{P_1} \cup \mathbf{P_2} \cup \mathbf{P_3} \cup \mathbf{P_4} \cup \dots \mathbf{P_{N-1}} \cup \mathbf{P_N}$

Value Functions

- Each person has a **subjective value function**, which gives a value to any subset of the cake
- More formally, it maps and subsets C ➡ [0,1] with SVF(C) = 1, SVF(ø) = 0
- We assume that all **SVF** are absolutely continuous, which mainly means that there are no "atoms", or no single points of cake worth any value
- We usually want our **SVFs** to be **sigma additive**, meaning that a value of a set, is the sum of the value of its parts
- We will assume our functions are **additive**

Proportionality

- Each person receives at least 1/N value using their own SVF
- More precisely, $\forall i: V_i(P_i) \geq 1/N$
- We can also have Weighted Proportional Division, where instead of each person receiving 1/N value, they receive it based upon their right over the cake
- More precisely, $\forall i: V_i(P_i) \geq w_i$

Envy Free

- Nobody envies any other person, i.e. no one would prefer someone else's slice more than their own
- Precisely, $\forall$ i,j : Vi(Pi) ≥ Vi(Pj)

Equitability

- Each person receives the same value of cake according to their own SVF
- Precisely, $\forall$ i,j : Vi(Pi) = Vj(Pj)

Divide and Choose

- Works only for 2 people
- Is Envy-Free
- If the SVFs are additive, then it is also Proportional
- Person 1 divides the cake into 2 pieces, each with value ½
- Person 2 then chooses from the 2 pieces, choosing the piece that they value more
- Person 1 remains with the other piece left over after Person 2 chooses

Fink Protocol
- Generalization of Divide and Choose for N people
- Ensures Proportionality for additive functions
- Online algorithm with N^3 / 3 pieces
- For 1 person, give them the whole cake
- For 2 people, employ Divide and Choose
- When you add the kth person, you split the portions of people [1..k-1] into k pieces, and person k chooses 1 pieces from each of people [1..k-1]'s pieces of their portions

Stromquist 3 Knife Procedure
- A Referee moves a sword from left to right over the cake
- Each player holds a knife at the point that divides the value of the cake to the right of the sword in half
- Any player calls the Ref to stop the sword when the value on the left of the sword is the same as each of the other pieces for them
- We then cut the cake using the knife in the middle of the 2 player knives

Implication

Number of People	Valuations	EF implied PR	PR implies EF
2	Additive	Yes	Yes
2	General	No	Yes
3+	Additive	Yes	No
3+	General	No	No

Pareto Efficiency (PE)
- If something is Pareto Efficient then the state cannot be changed without someone becoming less happy
- Very formally, if □ is a partition,
- □ is PE if $\exists !$ A partition □ | $\forall\ i : V^{\square}_i(P^{\square}_i) \geq V^{\square}_i(P^{\square}_i)$
- The simplest PE partition is just one person taking the entire cake

PEPR and PEEF Division
- A PEPR is a PE and PR division, and one always exists for N people
- A trivial example is if one person gets all the cake
- Interestingly, these cannot be found by any finite algorithm
- A PEEF is a PE and EF division, and always exists for N people if the SVFs are additive

- Existence is based by Weller's Theorem

Variants and Interesting Facts
- Cake can be discrete, and/or homogenous
- Fair Pie Cutting is when C is a circle, and the cuts must be radial
- If **SVFs** are not known to all, the Game Theory can model the effects of the lack of information
- MANY other ways to define "fairness"
- Problem is mostly studied for the cases of < 4 people, meaning a general solution for **N** people is unknown for general value functions
- Some problems don't have any **FPTAS** (Fully Polynomial Time Approximation Scheme) , meaning they can't even be approximated quickly

Applications
- Divorce settlement (discrete cake)
- Frequency/Spectrum allocation
- Land division
    - Some variants also allow land to be distributed so all new land is connected to its owner's land
- Division of chores (if items are undesirable)